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SORTING IN ONE ROUND

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ABSTRACT

Given an ordered set of n elements whose order is not known to us, it is shown that by making slightly more than $cn^{3/2}$ simultaneous comparisons almost all comparisons can be deduced by direct implications. It is also shown that this result is essentially best possible, and that if n is large, almost any choice of $cn^{3/2}$ comparisons will yield almost all comparisons by direct implications.

The paper is motivated by a *one round sorting problem* due to Pavol Hell [6]. Suppose we are given n objects in a linear order unknown to us. We choose q questions, that is pairs, whose order we wish to be revealed. Suppose no matter what answers we get, we can deduce at least $q+g$ comparisons. Then g is our *gain*: we deduce more comparisons than we are given. Pavol Hell asked for a distribution of the questions which maximizes our guaranteed gain. Another aim could be to maximize g/q or to minimize q under the condition that $q+g=\frac{1}{2}n^2+o(n^2)$, that is we learn almost all the answers. It is also often sensible to restrict our gain to *direct implications*: to comparisons we can deduce from at most two answers. Then the main problem is to find as few questions as possible so that we learn almost all comparisons as direct implications of the answers, no matter what the hidden order is. In this paper we shall concentrate on some variants of this last problem.

Though we are pointing out the obvious, we remark that the essence of the one round sorting problem is that the questions have to be distributed *before* any of the answers are obtained. It is clear that one needs at least $\log_2(n!)$ questions to determine a linear order completely. It is almost equally obvious that with $c\log_2(n!)$ questions one can determine the order. However, the questions have to be asked in several rounds and at each round the questions are distributed

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according to the answers received up to that point. Our problem bears a superficial resemblance to the restricted sorting problem discussed by several authors, including Batcher [1], Floyd and Knuth [5], Van Voorhis [8] and Knuth [7, p. 226–229]. However, in the restricted sorting problem the objects we compare may be interchanged so the subsequent questions do depend on the order. Note also that the complete order cannot be determined in one round unless we ask *all* $\binom{n}{2}$ questions. Thus we shall never require to learn more than almost all of the comparisons.

Before embarking on the essential part of the paper, we note that Häggkvist and Hell have shown independently that by asking $2n^{5/3}\log n$ appropriate questions, we may be guaranteed a gain of more than cn^2 pairs. We shall show that considerably more is true, namely that somewhat more than $cn^{3/2}$ questions can guarantee that almost every pair will be a direct implication. We shall show also that g/q can be as large as $cn^{1/2}$. Our paper concludes with some results concerning k-step implications.

Given a graph $G = (V, E)$, let $\vec{E}$ be an orientation of the edges. This orientation is said to be *consistent* or *acyclic* if the oriented graph $\vec{G} = (V, \vec{E})$ contains no directed cycle. Equivalently, $\vec{E}$ is an acyclic orientation if there is a linear order $<$ on V such that $xy \in E$ implies $x < y$. We say that $(x, z) \in V^{(2)}$ is a *direct implication* in an orientation $\vec{E}$ if there is a $y \in V$ for which $xy, yz \in \vec{E}$.

Let p be a prime (or a prime power) and put $n = p^2 + p + 1$. The following graph G_0 was constructed by Erdös and Rényi [4] (see also [2, p. 314]). The vertex set is the set of points of the projective plane $\mathrm{PG}(2, p)$ over the field of order p and a point (a, b, c) is joined to all the points on its polar with respect to the conic $x^2 + y^2 + z^2 = 0$. Thus (a, b, c) and (α, β, γ) are joined iff $a\alpha + b\beta + c\gamma = 0$.

THEOREM 1. *The graph G_0 has $n = p^2 + p + 1$ vertices, $\frac{1}{2}p(p+1)^2 \sim \frac{1}{2}n^{3/2}$ edges and every orientation of G_0 contains at least $\frac{1}{10}n^2(1 + o(1))$ direct implications.*

PROOF. The first two assertions are trivial. Furthermore, clearly every vertex has degree p or $p + 1$ and the graph contains no quadrilateral. (In fact, this is exactly the most interesting property of this graph, see [2, p. 314].)

We shall estimate the number of direct implications by studying the pentagons (5-cycles) in this graph. Let $x_2x_3 \in E(G_0)$. Let x_4 be a neighbour of x_3 but not of x_2. We have at least $p - 2$ choices for x_4. Now pick a neighbour x_1 of x_2 which is not a neighbour of x_3 or x_4. There are at least $p - 3$ choices for x_1. Then x_1 and x_4 must have exactly one common neighbour, which has to be distinct from x_2 and

x_3. In this way we have constructed a pentagon $x_1x_2\cdots x_5$. Since in constructing this pentagon we started with an arbitrary edge, we see that G has at least

$$\frac{1}{5}\frac{1}{2}p(p+1)^2(p-2)(p-3)\sim\frac{1}{10}p^5$$

pentagons.

Now let us estimate the number of pentagons in which a given pair $ab\in V^{(2)}$ can appear as a diagonal x_1x_3 of a pentagon $x_1x_2x_3x_4x_5$. Since x_1 and x_3 have at most one common neighbour, we have at most one choice for x_2. Then x_4 can be chosen in at most $p-1$ ways, and once again we have at most one choice for x_5. Hence at most $p-1$ pentagons contain ab as a diagonal.

Now it is easily checked that every orientation of a pentagon gives at least one direct implication. Consequently no matter how we orient G_0, we obtain at least

$$\frac{1}{10}p^5p^{-1}(1+o(1))=\frac{1}{10}p^4(1+o(1))=\frac{1}{10}n^2(1+o(1))$$

direct implications, as claimed. □

By a more careful and also more cumbersome argument the constant $\frac{1}{10}$ could be improved somewhat. However, as we shall see later, it cannot be improved to $\frac{1}{2}$. Now we come to the main result of the paper.

THEOREM 2. *Given $\varepsilon>0$ there is a constant $C(\varepsilon)$ such that for every n there exists a graph of order n having at most $C(\varepsilon)n^{3/2}$ edges whose every consistent orientation contains at least $\binom{n}{2}-\varepsilon n^2$ direct implications.*

In fact we shall prove considerably more, namely that almost every (a.e.) graph in a certain probability space $\mathcal{G}(n, P(\text{edge})=p)$ has the required properties, provided n is sufficiently large. Given $0<p<1$ and a natural number n, we write $\mathcal{G}(n, P(\text{edge})=p)$ for the discrete probability space consisting of all graphs on a fixed set V of n labelled vertices in which the probability of a fixed graph G_0 with m edges is

$$P(\{G_0\})=p^m(1-p)^{\binom{n}{2}-m}.$$

Equivalently: the edges are selected independently and with probability p. (See [3, ch. VII] for this model and for basic results on random graphs.) Our proof is based on the following two lemmas. Since the assertions of these lemmas concern a.e. graph, in the proofs we may and will assume that n is sufficiently large.

LEMMA 3. *Let α, β, η and C be positive constants with $\eta<1$. Put $p=Cn^{\eta-1}$.*

Then a.e. $G \in \mathcal{G}(n, P(\text{edge}) = p)$ *is such that for every* $W \subset V(G)$, $|W| = m \geqq \alpha n$ *the set*

$$Z_W = \left\{ x \in V - W : |\Gamma(x) \cap W| \leqq \frac{2}{3} pm \right\}$$

has at most βn *elements.*

PROOF. Let $W \subset V(G)$, $|W| = m \geqq \alpha n$ and $x \in V - W$ be fixed. A weak form of the De Moivre–Laplace theorem implies that for some constant $\gamma > 0$ we have

$$P(x \in Z_W) = p_0 \leqq e^{-\gamma n^{\eta}}.$$

Hence, rather crudely,

$$P(|Z_W| \geqq \beta n) \leqq 2^n p_0^{\beta n} \leqq 2^n e^{-\beta\gamma n^{1+\eta}} \to 0,$$

proving the lemma. □

LEMMA 4. *Let* γ, δ, η *and* C *be positive constants satisfying* $\eta < 1$ *and* $e^{C\gamma} > e/\delta$. *Put* $p = Cn^{\eta-1}$. *Then* a.e. *graph* $G \in \mathcal{G}(n, P(\text{edge}) = p)$ *is such that for every* $U \subset V(G)$, $|U| = u = \gamma n^{1-\eta}$ *the set*

$$T_U = \{x \in V - U : \Gamma(x) \cap U = \varnothing\}$$

has at most δn *elements.*

PROOF. For fixed U and $x \in V - U$ we have

$$P(x \in T_U) = (1-p)^u \leqq e^{-pu} \leqq e^{-C\gamma}.$$

Hence for a fixed set U:

$$P(|T_U| \geqq \delta n) \leqq \binom{n}{\delta n} e^{-C\gamma\delta n} \leqq \left(\frac{e}{\delta}\right)^{\delta n} e^{-C\gamma\delta n}.$$

Since there are less than $n^{n^{1-\eta}}$ choices for U,

$$P(|T_U| \geqq \delta n \text{ for some } U) \leqq n^{n^{1-\eta}} \exp\{-\delta n(C\gamma - \log(e/\delta))\}.$$

The conditions on the constants imply that this tends to 0 so the lemma follows. □

PROOF OF THEOREM 2. Choose two constant $C(\varepsilon)$ and C satisfying $C(\varepsilon) > C > (9 \log(3e/\varepsilon)/\varepsilon)^{\frac{1}{2}}$. Put $p = Cn^{-1/2}$. We shall show that a.e. $G \in \mathcal{G}(n, P(\text{edge}) = p)$ has the required property.

Write $\eta = \frac{1}{2}$, $k = \lceil 4/\varepsilon \rceil$, $\alpha = 1/(2k)$, $\beta = \varepsilon/(8k)$, $\gamma = C/(2k)$ and $\delta = \varepsilon/3$. Then a.e. graph satisfies the conclusions of Lemmas 3 and 4 and has at most $C(\varepsilon)n^{3/2}$ edges. We shall show that every such graph G_0 will do for the theorem. Having been given an arbitrary consistent orientation of G_0, let us relabel the vertices $x_1, x_2, \cdots, x_n$ of G_0 in such a way that every edge x_ix_j, $i > j$, is oriented from i to j. Partition $V(G_0)$ into k consecutive blocks of roughly equal size: $V(G) = \bigcup_1^k W_i$, $\max W_i + 1 = \min W_{i+1}$ and $\lfloor n/k \rfloor \leqq |W_i| \leqq \lceil n/k \rceil$.

Pick a block W_i, $i \geqq 3$, and write $W = W_{i-1}$. Then by Lemma 3 we have $|Z_W| \leqq \beta n$ so

$$|Z_W \cap W_i| \leqq \beta n.$$

Each $x \in W_i - Z_W$ is joined to a set U_x of at least $\frac{2}{3}p\,|W| \geqq \gamma n^{1/2}$ vertices in W. By Lemma 4 at most δn vertices of G_0 are not joined to some vertices in U_x. Hence for this x there are at most $2\lceil n/k \rceil + \delta n < \frac{6}{7}\varepsilon n$ vertices $y \in \bigcup_{j=1}^{i} W_j$ such that xy is not a direct implication. Consequently there are at most $\beta n^2 + \lceil n/k \rceil \frac{6}{7}\varepsilon n < \varepsilon n^2/k$ edges of the form x_jx_k, $x_j \in W_i$, $k < j$, for which x_jx_k is not a direct implication. As there are k blocks W_i, the proof is complete. □

From what we have said so far it is not clear that Theorems 1 and 2 cannot be improved greatly. We do not know that there is no graph of size $O(n^{3/2})$ whose every consistent orientation misses only $o(n^2)$ direct implications. Now we show not only that this is the case, but also that the dependence of $C(\varepsilon)$ on ε in Theorem 2 is essentially best possible. The factor $(\log(1/\varepsilon))^{1/2}$ is very likely to be an error term due to the probabilistic method.

THEOREM 5. (i) *Every graph G of order n and size $m \leqq 2^{-5/2}n^{3/2}$ has a consistent orientation with at most $\binom{n}{2} - \frac{1}{16}n^2$ direct implications.*

(ii) *Let C and ε be positive constants satisfying $\varepsilon < 2^{-14}C^{-2}$. Then if n is sufficiently large, every graph G of order n and size $m \leqq Cn^{3/2}$ has a consistent orientation with at most $(\frac{1}{2} - \varepsilon)n^2$ direct implications.*

PROOF. (i) Let $x_1, x_2, \cdots, x_n$ be the vertices of G such that $d(x_i) \leqq d(x_{i+1})$. Then $d(x_i) \leqq 2m/([n/2]+2)$ for $i \leqq [n/2] - 1$. Orient each edge x_ix_j of G from i to j if $i < j$. Let us say that a direct implication $\overrightarrow{x_ix_k}$ *belongs to* x_j if $i < j < k$ and $x_ix_j, x_jx_k \in E(G)$. Then at most $d(x_j)^2/4$ direct implications belong to x_j. Consequently the number of direct implications in $\{x_1, x_2, \cdots, x_{[n/2]}\}$ is at most

$$\sum_{j=2}^{\lceil n/2 \rceil - 1} d(x_j)^2/4 \leqq (\lceil n/2 \rceil - 2)(m/(\lfloor n/2 \rfloor + 2))^2/4$$

$$< \binom{\lceil n/2 \rceil}{2} - \frac{1}{16}n^2.$$

(ii) For simplicity we shall omit the integrality signs — this clearly does not affect the argument.

Put $\delta = 2^5\varepsilon$. Let W be an arbitrary set of $n/2$ vertices. Since $G[W]$ has at most $Cn^{3/2}$ edges, there is a subset $Z \subset W$ with δn vertices and at most

$$(2\delta)^2 Cn^{3/2} = (4C\delta^{1/2})(\delta n)^{3/2} \leqq 2^{-5/2}(\delta n)^{3/2}$$

edges. Hence there are disjoint subsets $Z_1, Z_2, \cdots, Z_k$, $k = 1/(2\delta)$, $|Z_i| = \delta n$, such that each $G[Z_i]$ has fewer than $2^{-5/2}(\delta n)^{3/2}$ edges. Order the vertices of G so that each Z_i is a block, that is no vertex $x \notin Z_i$ is both preceded and succeeded by vertices of Z_i. Then no pair yz, $y, z \in Z_i$ can be implied by edges not in Z_i. Consequently by part (i) in each $G[Z_i]$ at least $\frac{1}{16}(\delta n)^2$ implications are missing. Hence in G altogether at least

$$\frac{1}{2\delta} \cdot \frac{1}{16}(\delta n)^2 = 2^{-5}\delta n^2 = \varepsilon n^2$$

direct implications are missing. □

Note that a head-on attack on (i) or (ii) making use of large empty subgraphs of G gives only $O(n^{3/2})$ missing implications since by Turán's theorem $O(n^{3/2})$ edges guarantee only an empty graph of order $O(n^{1/2})$.

Slightly more elaborate and more tedious versions of the proofs above give us analogous results for k-step implications. We leave the details to the reader. A *k-step implication* in an oriented graph is a pair of vertices (x, y) for which there is a directed path of length k from x to y. Thus a direct implication is exactly a 2-step implication.

THEOREM 6. (i) *Given $\varepsilon > 0$ there is a constant $C > 0$ such that with $p = Cn^{1/k-1}$* a.e. *graph $G \in \mathscr{G}(n, P(\text{edge}) = p)$ contains at least $\binom{n}{2} - \varepsilon n^2$ k-step implications in every consistent orientation of G. In particular, if $C(n) \to \infty$ then there is a graph of order n and size at most $C(n)n^{1+1/k}$ whose every consistent orientation contains $\binom{n}{2} + o(n^2)$ k-step implications.*

(ii) *Given $C > 0$ there is an $\varepsilon > 0$ such that if n is sufficiently large then every graph of order n and size $m \leqq Cn^{1+1/k}$ has a consistent orientation with at most $(\frac{1}{2} - \varepsilon)n^2$ k-step implications.* □

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